METHODS OF FINDING PARTICULAR INTEGRAL

Consider the differential equation

$$P_0(x)\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = X \longrightarrow (1)$$

Let, $D \equiv \frac{d}{dx}$.

Then $D^n \equiv \frac{d^n}{dx^n}$.

i.e. The equation (1) becomes

$$\left[P_0(x)D^n + P_1(x)D^{n-1} + \cdots + P_{n-1}(x)D + P_n(x)\right]y = X$$

or, $f(D)y = X$, where $f(D) = P_0(x)D^n + \cdots + P_n(x)$.

Then the particular integral will be

$$y_p = \frac{1}{f(D)}\cdot X$$

Case-I

When X is an exponential function.

i.e. $X = e^{ax}$.

Then $y_p = \frac{1}{f(D)}\cdot e^{ax}$

$= \frac{e^{ax}}{f(a)}$ [Provided $f(a) \neq 0$]

Example:→ Evaluate : $\frac{1}{D^2-5D+6}\cdot e^{4x}$.

$\therefore y_p = \frac{1}{D^2-5D+6}\cdot e^{4x} = \frac{e^{4x}}{4^2-5\times4+6} = \frac{1}{2}e^{4x}$.

Case-II

When $X = e^{ax}$ and $f(a) = 0$.

Then $y_p = \frac{1}{f(D)}e^{ax} = \frac{x}{f'(D)}e^{ax} = \frac{x}{f'(a)}e^{ax}$, provided $f'(a) \neq 0$.

Method-2 $y_p = \frac{1}{f(D)}\cdot e^{ax} = e^{ax}\cdot\frac{1}{f(D+a)}\cdot 1$

Example:→ Evaluate : $\frac{1}{D^2-5D+6}\cdot e^{3x}$

$\frac{1}{D^2-5D+6}\cdot e^{3x} = \frac{x}{2D-5}\cdot e^{3x}$ [$\because 3^2-5\times3+6=0$]

$= \frac{x}{6-5}e^{3x} = xe^{3x}$

Case - III

When X is an algebraic function.

i.e. $X = x^m$.

Then $y_p = \frac{1}{f(D)} \cdot x^m = \{f(D)\}^{-1} \cdot x^m$

And then $\{f(D)\}^{-1}$ is expanded by the negative binomial formula.

$(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$

$(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$

$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$

$(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$

Example :- 1. Evaluate : $\frac{1}{(D-1)^2} \cdot x^3$.

$$\begin{aligned}
\frac{1}{(D-1)^2} \cdot x^3 &= (D-1)^{-2} \cdot x^3 \\
&= (1-D)^{-2} \cdot x^3 \\
&= (1 + D + D^2 + D^3 + D^4 + \cdots)x^3 \\
&= x^3 + Dx^3 + D^2x^3 + D^3x^3 + D^4x^3 + \cdots \\
&= x^3 + 3x^2 + 6x + 6 + 0 \\
&= x^3 + 3x^2 + 6x + 6.
\end{aligned}$$

$\left[\because D \equiv \frac{d}{dx}.\ D^4x^3 = \frac{d^4(x^3)}{dx^4} = 0,\ D^5x^3 = D^6x^3 = \cdots = 0\right]$

2. Evaluate : $\frac{1}{D^3+D} \cdot x^2$

$$\begin{aligned}
\frac{1}{D^3+D} \cdot x^2 &= \frac{1}{D(D^2+1)} \cdot x^2 \\
&= \frac{1}{D(1+D^2)} x^2 \\
&= \frac{1}{D}(1+D^2)^{-1} x^2 \\
&= \frac{1}{D}(1 - D^2 + D^4 - D^6 + \cdots)x^2 \\
&= \frac{1}{D}(x^2 - 2) \quad [\because D^3x^2 = D^4x^2 = \cdots = 0] \\
&= \frac{x^3}{3} - 2x \quad \left[\frac{1}{D} = \text{anti-derivative} = \text{integration}\right]
\end{aligned}$$

3. Evaluate : $\frac{1}{D^2-5D+6} \cdot x^2$

$$\begin{aligned}
\frac{1}{D^2-5D+6} \cdot x^2 &= \frac{1}{6\left(1 + \frac{D^2-5D}{6}\right)} \cdot x^2 \\
&= \frac{1}{6}\left(1 + \frac{D^2-5D}{6}\right)^{-1} \cdot x^2 \\
&= \frac{1}{6}\left(1 - \frac{D^2-5D}{6} + \frac{(D^2-5D)^2}{6^2} - \cdots\right)x^2 \\
&= \frac{1}{6}\left(1 - \frac{D^2-5D}{6} + \frac{1}{36}(D^4 - 10D^3 + 25D^2) - \cdots\right)x^2 \\
&= \frac{1}{6}\left[x^2 - \frac{1}{6}(2 - 10x) + \frac{25}{18}\right]
\end{aligned}$$

Case-IV

When $X = \sin ax$ or $X = \cos ax$.

In this case $y_p = \frac{1}{f(D)} \cdot \sin ax$

$$= \frac{\psi(D)}{\phi(D^2)} \cdot \sin ax$$

Then we replace D^2 by $(-a^2)$ and if $\phi(-a^2) \neq 0$

then $y_p = \frac{\psi(D)}{\phi(-a^2)} \sin ax$.

Example :→ 1. Evaluate : $\frac{1}{D^2+4} \sin x$

$$\frac{1}{D^2+4} \sin x = \frac{1}{-1^2+4} \cdot \sin x = \frac{1}{3} \sin x$$

2. Evaluate : $\frac{1}{D^2-5D+6} \cdot \cos 4x$

$$= \frac{1}{-4^2-5D+6} \cos 4x$$

$$= \frac{1}{-5D-10} \cos 4x$$

$$= -\frac{1}{5(D+2)} \cos 4x$$

$$= -\frac{(D-2)}{5(D+2)(D-2)} \cos 4x$$

$$= -\frac{1}{5} \frac{(D-2)}{(D^2-4)} \cos 4x$$

$$= -\frac{1}{5} \frac{(D-2)}{(-4^2-4)} \cos 4x$$

$$= \frac{1}{60} (D-2) \cos 4x$$

$$= \frac{1}{60} (-4\sin 4x - 2\cos 4x)$$

$$= -\frac{1}{30} (2\sin 4x + \cos 4x)$$

Case-V

When $X = \sin ax$ or $X = \cos ax$ with $\phi(-a^2) = 0$.

; Where $y_p = \frac{1}{f(D)} \cdot \sin ax = \frac{\psi(D)}{\phi(D^2)} \sin ax$.

Then let, $P = \frac{\psi(D)}{\phi(D^2)} \cos ax$ and $Q = \frac{\psi(D)}{\phi(D^2)} \cdot \sin ax$.

Then $P + iQ = \frac{\psi(D)}{\phi(D^2)} e^{iax}$.

In this case we calculate $\frac{\psi(D)}{\phi(D^2)} e^{iax}$.

Let, $\frac{\psi(D)}{\phi(D^2)} e^{iax} = u(x) + iv(x)$

Then $P = u(x)$ and $Q = v(x)$.

Example :→ Evaluate: $\frac{1}{D^2+a^2}\sin ax$, $\frac{1}{D^2+a^2}\cos ax$.

Here $-a^2+a^2=0$

So, $\frac{1}{D^2+a^2}(\cos ax+i\sin ax)$

$= \frac{1}{D^2+a^2}e^{iax}$

$= \frac{1}{(D+ia)(D-ia)}\cdot e^{iax}$

$= \frac{1}{2ia}\cdot\frac{1}{D-ia}e^{iax}$

$= \frac{1}{2ia}\frac{e^{iax}}{(D+ia)-ia}\cdot 1$

$= \frac{e^{iax}}{2ia}\cdot\frac{1}{D}\cdot 1$

$= \frac{e^{iax}}{2ia}x$

$= \frac{x}{2ia}(\cos ax+i\sin ax)$

$= \frac{x}{2a}(\sin ax - i\cos ax)$.

$= \frac{x\sin ax}{2a}+i\left(-\frac{x\cos ax}{2a}\right)$

$\therefore \frac{1}{D^2+a^2}\sin ax = -\frac{x\cos ax}{2a}$, and $\frac{1}{D^2+a^2}\cos ax = \frac{x\sin ax}{2a}$.

Case-VI

When $X = e^{ax}\cdot v$, where v is any function of x.

In this case, $y_p = \frac{1}{f(D)}\cdot e^{ax}v$

$= e^{ax}\cdot\frac{1}{f(D+a)}v$

Example :→ 1. Evaluate : $\frac{1}{(D-1)^2}\cdot x^2e^{3x}$

$\frac{1}{(D-1)^2}\cdot x^2e^{3x} = e^{3x}\frac{1}{\{(D+3)-1\}^2}\cdot x^2$

$= e^{3x}\frac{1}{(D+2)^2}\cdot x^2$

$= \frac{e^{3x}}{2^2}\frac{1}{(1+\frac{D}{2})^2}\cdot x^2$

$= \frac{e^{3x}}{4}\left(1+\frac{D}{2}\right)^{-2}\cdot x^2$

$= \frac{e^{3x}}{4}\left(1-2\cdot\frac{D}{2}+3\cdot\frac{D^2}{2^2}-4\cdot\frac{D^3}{2^3}+\cdots\right)x^2$

$= \frac{e^{3x}}{4}\left(x^2-2x+\frac{3}{2}\right)$

$= \frac{e^{3x}}{8}(2x^2-4x+3)$

2. Evaluate : $\frac{1}{D^2-2D+4} \cdot e^x \cos x$

$$\frac{1}{D^2-2D+4} \cdot e^x \cos x$$

$$= \frac{1}{(D-2)^2} \cdot e^x \cos x$$

$$= e^x \frac{1}{\{(D+1)-2\}^2} \cdot \cos x$$

$$= e^x \cdot \frac{1}{(D-1)^2} \cos x$$

$$= e^x \cdot \frac{1}{D^2-2D+1} \cos x$$

$$= e^x \cdot \frac{1}{-1-2D+1} \cos x$$

$$= e^x \cdot \frac{1}{(-2D)} \cos x$$

$$= -\frac{e^x}{2} \cdot \sin x \quad [\frac{1}{D} = \text{integration}]$$

Case - VII

When $X = x \cdot v$, where v is other function of x.

In this case, $y_p = \frac{1}{f(D)} xv$

$$= x \cdot \frac{1}{f(D)} \cdot v - \frac{f'(D)}{\{f(D)\}^2} \cdot v$$

Example :→

$$\frac{1}{D^3-1} x \sin x$$

$$= x \cdot \frac{1}{D^3-1} \cdot \sin x - \frac{3D^2}{(D^3-1)^2} \cdot \sin x$$

$$= x \cdot \frac{1}{-D-1} \sin x - \frac{3(-1)}{(-D-1)^2} \sin x$$

$$= -x \cdot \frac{(D-1)}{(D+1)(D-1)} \sin x + \frac{3}{D^2+2D+1} \cdot \sin x$$

$$= -x \frac{D-1}{D^2-1} \sin x + \frac{3}{-1+2D+1} \sin x$$

$$= -x \frac{D-1}{-1-1} \sin x + \frac{3}{2D} \sin x$$

$$= \frac{x}{2} \cdot (D-1) \sin x + \frac{3}{2} (-\cos x)$$

$$= \frac{x}{2} (\cos x - \sin x) - \frac{3}{2} \cos x .$$

Example :→ Solve : $\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos 2x$

Solution :→

$$\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos 2x \longrightarrow (1)$$

Let, $y = e^{mx}$ be a trial solution of the corres-ponding homogeneous eqn.

Then the auxiliary eqn is

$$m^2 + 2 = 0$$
$$\Rightarrow m^2 = -2$$
$$\Rightarrow m = \pm\sqrt{2}\,i$$

∴ The complementary function is

C.F. $= y_c = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x$, where c_1 and c_2 are arbitrary constant.

Now the particular integral,

$$\text{P.I.} = y_p = \frac{1}{D^2+2}\cdot\left(x^2e^{3x} + e^x\cos 2x\right)$$

$$= \frac{1}{D^2+2}\cdot x^2e^{3x} + \frac{1}{D^2+2}\cdot e^x\cos 2x$$

$$= e^{3x}\frac{1}{(D+3)^2+2}\cdot x^2 + e^x\cdot\frac{1}{(D+1)^2+2}\cdot\cos 2x$$

$$= e^{3x}\frac{1}{D^2+6D+11}\cdot x^2 + e^x\frac{1}{D^2+2D+3}\cdot\cos 2x$$

$$= \frac{e^{3x}}{11}\cdot\left\{1 + \frac{D^2+6D}{11}\right\}^{-1}\cdot x^2 + e^x\frac{1}{-4+2D+3}\cos 2x$$

$$= \frac{e^{3x}}{11}\left\{1 - \frac{D^2+6D}{11} + \left(\frac{D^2+6D}{11}\right)^2 - \cdots\right\}x^2$$
$$+ e^x\cdot\frac{1}{2D-1}\cos 2x$$

$$= \frac{e^{3x}}{11}\left\{x^2 - \frac{2+12x}{11} + \frac{72}{121}\right\}$$
$$+ e^x\cdot\frac{2D+1}{4D^2-1}\cos 2x$$

$$= \frac{e^{3x}}{11}\left\{x^2 - \frac{12x}{11} + \frac{50}{121}\right\} + e^x\cdot\frac{(2D+1)}{-16-1}\cos 2x$$

$$= \frac{e^{3x}}{11}\left\{x^2 - \frac{12x}{11} + \frac{50}{121}\right\} - \frac{e^x}{17}(\cos 2x - 4\sin 2x)$$

∴ The general solution of the given differential equation is $y = y_c + y_p$

i.e. $y = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{e^{3x}}{121}\left\{11x^2 - 12x + \frac{50}{11}\right\}$
$+ \frac{e^x}{17}(4\sin 2x - \cos 2x).$

EXERCISES :→

1. Evaluate the following:

i) $\frac{1}{D^2-1}\, xe^x$

ii) $\frac{1}{(D-2)^2} \cdot x^3 e^{2x}$

iii) $\frac{1}{D^2-2D+4} \cdot e^x \cos^2 x$

2. Solve the following equations:

i) $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + (a^2+b^2)y = e^{px}$

ii) $(D^3+2D^2+D)y = e^{2x} + x^2 + x$

iii) $(D^3-3D^2+3D-1)y = xe^x + e^x$

iv) $\frac{d^2y}{dx^2} - y = x^2\cos x$

v) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

vi) $\frac{d^4y}{dx^4} - y = e^x \cos x$

vii) $(D^2-1)y = x\sin x + (1+x^2)e^x$

Answers :

1.

i) $\frac{1}{4}e^x\left(x^2 - x + \frac{1}{2}\right)$, ii) $\frac{1}{20}e^{2x}x^5$, iii) $\frac{1}{6}e^x - \frac{1}{2}e^x\cos 2x$

2.

i) $y = e^{-ax}(c_1\cos bx + c_2\sin bx) + \frac{e^{px}}{(a+p)^2+b^2}$

ii) $y = c_1 + (c_2+c_3x)e^{-x} + \frac{x}{6}(2x^2-9x+24) + \frac{1}{18}e^{2x}$

iii) $y = (c_1+c_2x+c_3x^2)e^x + \frac{1}{24}e^x(x^4+4x^3)$

iv) $y = c_1e^x + c_2e^{-x} + \frac{1}{2}(1-x^2)\cos x + x\sin x$

v) $y = (c_1+c_2x)e^x - (x\sin x + 2\cos x)e^x$

vi) $y = c_1e^x + c_2e^{-x} + (c_3\cos x + c_4\sin x) - \frac{1}{5}e^x\cos x$

vii) $y = c_1e^x + c_2e^{-x} + \frac{1}{12}xe^x(2x^2-3x+9) - \frac{1}{2}(x\sin x + \cos x)$

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